JOURNAL OF APPROXIMATION THEORY **89**, 118–132 (1997)
ARTICLE NO. AT963033

On the Sequence of Errors in Best Polynomial Approximation

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Communicated by András Kroó

Received October 16, 1995; accepted February 27, 1996

We consider the sequence of errors $(E_n(f))_n$ of best uniform approximation to a function $f \in C[-1, 1]$ by algebraic polynomials. It is shown that the regularity of f in subsets of $[-1, 1]$ implies certain conditions on the sequence $(E_n(f))_n$.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

For a given complex-valued function $f \in C[-1, 1]$ let

$$E_n(f) := \min_{p \in P_n} \|f - p\|_{[-1,1]} = \|f - p_n^*\|_{[-1,1]}$$

denote the error of the best uniform approximation $p_n^* = p_n^*(f)$ to f in the set P_n of algebraic polynomials of degree at most $n \in \mathbb{N}_0$. By the classical Weierstrass approximation theorem we know that $E_n(f) \searrow 0$. The construction of functions having prescribed error sequences was first treated by Bernstein [1].

Theorem A (Bernstein; cf. [4, p. 121]). *Let be given a sequence $(E_n)_n$ with $E_n \searrow 0$. Then there exists a function $f \in C[-1, 1]$ such that $E_n(f) = E_n$ for all $n \in \mathbb{N}_0$.*

Recently, Professor Gaier raised the question of whether the function f in Theorem A can be constructed such that f is not only continuous on $[-1, 1]$ but also regular in subsets of $[-1, 1]$. In the present paper we show that this is not possible for arbitrary sequences $(E_n)_n$ with $E_n \searrow 0$. It will turn out that the regularity of f in subsets of $[-1, 1]$ implies that there can not be too abrupt "jumps" in the error sequence $(E_n(f))_n$.

To state the results, let $(E_n)_n$ be a sequence with $E_n \searrow 0$ and

$$r := \limsup_{n \to \infty} E_n^{1/n}.$$

0021-9045/97 $25.00

In the following we denote by f an arbitrary function in $C[-1, 1]$ such that

$$E_n(f) = E_n \qquad \text{for all} \quad n \in \mathbb{N}_0.$$

Further, let $g(z) := \log |z + (z^2 - 1)^{1/2}|$ denote the Green's function of $[-1, 1]^c$ with pole at ∞ and $C_s := \{z\colon g(z) = \log(s)\}$, $s \geqslant 1$. For $s = 1$, we have $C_s = [-1, 1]$, while for $s > 1$ the level curve C_s is given by the boundary of an ellipse with foci at -1 and 1. It is well known ([9, p. 79]) that

$$\frac{1}{r} = \sup\,\{s\colon f \text{ is holomorphic in the open ellipse bounded by } C_s\}.$$

1.1. *Results for Real-Valued Functions*

THEOREM 1. *Let $r = 1$ and suppose that there exists a subsequence L of $\mathbb{N}$ with $\lim_{n \in L} E_n^{1/n} = 1$ such that*

$$\eta := \lim_{n \in L} \frac{E_{n+1}}{E_n} < 1,$$

and

$$\lambda := \sup_{\alpha \in (0, 1)} \; \limsup_{n \in L} \frac{E_n}{E_{[\alpha n]}} > 0.$$

Let f be real-valued and regular in the open set $A \subset [-1, 1]$.

Then the following properties hold:

(a) *If $\eta = 0$ and $\mu_{[-1, 1]}$ denotes the equilibrium distribution of $[-1, 1]$, then we have*

$$\mu_{[-1, 1]}(A) \leqslant 1 - \lambda.$$

(b) *If $\lambda = 1$, then $A = \varnothing$.*

From this result one can derive the following Hadamard-type gap theorem:

THEOREM 2. *Let $r = 1$ and suppose that there exists a subsequence $(n_k)_{k \in \mathbb{N}_0}$ of $\mathbb{N}$ such that for all $k \in \mathbb{N}_0$,*

$$\frac{n_{k+1}}{n_k} \geqslant \rho > 1$$

and

$$E_n = E_{n_k+1} \qquad \textit{for all} \quad n_k+1 \leqslant n \leqslant n_{k+1}.$$

Let f be real-valued.

Then the following properties hold:

(a) *If* $\liminf_{k\to\infty} E_{n_k}^{1/n_k} < 1$, *then f has no regular point in* $[-1, 1]$.

(b) *If* $\lim_{k\to\infty} E_{n_k}^{1/n_k} = 1$ *and* $E_n = O(n^{-\beta})$ *for some* $\beta > 0$, *then f has no regular point in* $[-1, 1]$.

In case $r \in (0, 1)$, Theorem 5 states an analogous result without needing the O-condition which is assumed in part (b) of Theorem 2. The method of proof of Theorem 5 cannot be applied if $r = 1$.

1.2. *Results for Complex-Valued Functions*

For complex-valued functions $f = \operatorname{Re} f + i \operatorname{Im} f \in C[-1, 1]$, it is easy to see that f is regular at some point $x_0 \in [-1, 1]$ if and only if its real part $\operatorname{Re} f$ and imaginary part $\operatorname{Im} f$ (defined for $x \in [-1, 1]$) are both regular at x_0. Thus, Theorems 1 and 2 may be applied to the error sequences of $\operatorname{Re} f$ and $\operatorname{Im} f$ to obtain estimates on the "size" of sets where f can be regular.

The following estimates are based on the behaviour of $(E_n)_n = (E_n(f))_n$.

THEOREM 3. *Let $r = 1$ and suppose that there exists a subsequence L of $\mathbb{N}$ with* $\lim_{n\in L} E_n^{1/n} = 1$ *such that*

$$\eta := \lim_{n\in L} \frac{E_{n+1}}{E_n} = 0,$$

and

$$\lambda := \sup_{\alpha\in(0,1)} \limsup_{n\in L} \frac{E_n}{E_{[\alpha n]}} > 0.$$

Let f be regular in the open set $A \subset [-1, 1]$.

Then, if $\mu_{[-1,1]}$ denotes the equilibrium distribution of $[-1, 1]$, we have

$$\mu_{[-1,1]}(A) \leqslant 1 - \frac{\lambda}{2^{1/2}}.$$

If E_{n+1}/E_n tends to zero very rapidly for some subsequence L, it is not necessary to consider the behaviour of the foregoing errors $E_{[\alpha n]}$:

THEOREM 4. *Let $r \in (0, 1]$ and suppose that there exists a subsequence L of $\mathbb{N}$ such that*

$$\lim_{n \in L} E_n^{1/n} = r \qquad \text{and} \qquad \limsup_{n \in L} E_{n+1}^{1/n+1} < r.$$

Then f has no regular point on $C_{1/r}$.

Remark. In Theorem 4 it is a necessary assumption that $\lim_{n \in L} E_n^{1/n} = r$. It is possible to construct a sequence $(E_n)_n$ with $r = 1$ and a corresponding function f which is regular in $(-1, 1)$, such that for a suitable subsequence L we have

$$\lim_{n \in L} E_{n+1}^{1/n+1} < \lim_{n \in L} E_n^{1/n} < r.$$

In case f is regular on $[-1, 1]$, i.e., $r < 1$, one can derive from Theorem 4 the following Hadamard-type gap theorem.

THEOREM 5. *Let $r \in (0, 1)$ and suppose that there exists a subsequence $(n_k)_{k \in \mathbb{N}_0}$ of $\mathbb{N}$ such that for all $k \in \mathbb{N}_0$*

$$\frac{n_{k+1}}{n_k} \geqslant \rho > 1$$

and

$$E_n = E_{n_k+1} \qquad \text{for all} \quad n_k + 1 \leqslant n \leqslant n_{k+1}.$$

Then f has no regular point on $C_{1/r}$.

The proof of Theorem 4 and Theorem 5 is based on methods of harmonic majorization and can be extended to the case of uniform approximation on more general compact sets in the complex plane.

2. PROOFS

Proof of Theorem 1. Since f is real-valued, there exists for each $n \in \mathbb{N}_0$ a set A_n of alternation points

$$-1 \leqslant x_{n,1} < \cdots < x_{n,n+2} \leqslant 1$$

of the error function $f - p_n^*$, i.e.,

$$(f - p_n^*)(x_{n,j}) = \pm(-1)^j E_n \qquad \text{for all} \quad j \in \{1, ..., n+2\}.$$

The set A_n defines an extremal signature for $f-p_n^*$ (cf. [6, p. 76]) which consists of exactly $n+2$ points. We put $w_n(x):=\prod_{j=1}^{n+2}(x-x_{n,j})$ and $t_n:=(\sum_{j=1}^{n+2}(1/|w_n'(x_{n,j})|))^{-1}$. By

$$\mu_n(x_{n,j}):=\frac{t_n}{|w_n'(x_{n,j})|}, \qquad j\in\{1, ..., n+2\},$$

a discrete measure μ_n of total mass one is defined on the set A_n, which is associated with the extremal signature on A_n ([6, p. 78]). From properties of extremal signatures ([6, p. 76]) it is known that

$$\sum_{j=1}^{n+2}\mu_n(x_{n,j})\,(f-p_n^*)(x_{n,j})\,p(x_{n,j})=0$$

holds for every $p\in P_n$.

1. First, we consider the measures μ_n, $n\in L$.

Let be given a Borel set $B\subset[-1, 1]$ with $b:=\mu_{[-1,1]}(B)\in(0, 1)$.

Let m_n denote the number of points in $B\cap A_n$ and let $b_n:=\mu_n(B)$.

In ([2, Theorem 1]) it was proved that certain subsequences of the unit counting measures of any $n+2$ Fekete points of $\{x\in[-1, 1]: |(f-p_n^*)(x)|=E_n\}$ converge to $\mu_{[-1,1]}$ in the sense of weak convergence. Following the proof of Theorem 1 in [2] one can see that the same holds for the subsequence L of the unit counting measures of A_n. Hence, we obtain

$$\lim_{n\in L}\frac{m_n}{n+2}=b=\mu_{[-1,1]}(B). \tag{2}$$

We have (cf. for example [2, p. 362])

$$t_n=\min_{p\in P_n}\|x^{n+1}-p(x)\|_{A_n}\leqslant\min_{p\in P_n}\|x^{n+1}-p(x)\|_{[-1,1]}=\frac{1}{2^n}.$$

Since A_n contains an extremal signature for $f-p_n^*$, it follows that ([6, p. 78])

$$E_n=\min_{p\in P_n}\|f-p\|_{[-1,1]}=\min_{p\in P_n}\|f-p\|_{A_n},$$

and we can follow an argument of Kroó and Saff ([3, Lemma 2.3]) to obtain

$$\gamma_n:=\frac{t_n}{1/2^n}\geqslant\frac{E_n-E_{n+1}}{E_n+E_{n+1}}=\frac{1-E_{n+1}/E_n}{1+E_{n+1}/E_n}, \tag{3}$$

and therefore

$$\gamma := \liminf_{n \in L} \gamma_n \geqslant \frac{1-\eta}{1+\eta} > 0.$$

Let V_n denote the $n+2$ point discriminant of the set A_n, i.e.,

$$V_n := \left(\prod_{j=1}^{n+2} \prod_{k=j+1}^{n+2} |x_{n,j} - x_{n,k}| \right)^2 = \prod_{j=1}^{n+2} |w'_n(x_{n,j})|.$$

We show that

$$t_n \leqslant \left(V_n \frac{b_n^{m_n}(1-b_n)^{n+2-m_n}}{m_n^{m_n}(n+2-m_n)^{n+2-m_n}} \right)^{1/(n+2)}.$$

To prove this, let n be fixed and consider the problem of finding the supremum of

$$t = t(\xi) = \left(\sum_{j=1}^{n+2} \frac{1}{\xi_j} \right)^{-1}$$

among all $\xi = (\xi_1, ..., \xi_{n+2})$ satisfying the restrictions

$$\xi_j > 0 \qquad \text{for all} \quad j \in \{1, ..., n+2\},$$

$$V(\xi) := \prod_{j=1}^{n+2} \xi_j = V_n \qquad \text{and} \qquad \mu(\xi) := t(\xi) \sum_{j \in J} \frac{1}{\xi_j} = b_n,$$

where $J \subset \{1, ..., n+2\}$ is an arbitrary subset of indices consisting of m_n points. If $\xi_j \to 0$ or $\xi_j \to \infty$ for some $j \in \{1, ..., n+2\}$, we obtain $t(\xi) \to 0$, and therefore a global maximum of t must be attained for some point ξ^*. By the theorem of Lagrange, there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\frac{\partial t}{\partial \xi_j}(\xi^*) + \lambda_1 \frac{\partial V}{\partial \xi_j}(\xi^*) + \lambda_2 \frac{\partial \mu}{\partial \xi_j}(\xi^*) = 0 \qquad \text{for all} \quad j \in \{1, ..., n+2\}.$$

A simple computation gives

$$\xi_j^* = \frac{t(\xi^*) m_n}{b_n}, \qquad \text{for all} \quad j \in J$$

and

$$\xi_j^* = \frac{t(\xi^*)(n+2-m_n)}{1-b_n}, \qquad \text{for all} \quad j \in \{1, ..., n+2\} \backslash J.$$

Thus, we have

$$V_n = \prod_{j=1}^{n+2} \xi_j^* = \left(\frac{t(\xi^*)\, m_n}{b_n}\right)^{m_n} \left(\frac{t(\xi^*)(n+2-m_n)}{1-b_n}\right)^{n+2-m_n},$$

which yields

$$\begin{aligned} t_n = t(|w_n'(x_{n,1})|, ..., |w_n'(x_{n,n+2})|) &\leqslant t(\xi^*) \\ &= \left(V_n \frac{b_n^{m_n}(1-b_n)^{n+2-m_n}}{m_n^{m_n}(n+2-m_n)^{n+2-m_n}}\right)^{1/(n+2)}. \end{aligned} \tag{4}$$

If Δ_{n+2} denotes the discriminant of the $n+2$ Fekete points of $[-1, 1]$, then $V_n \leqslant \Delta_{n+2}$, and by ([5, p. 422]) it follows that

$$\Delta_{n+2} \sim \text{const}(n+2)^{n+2+1/4} \frac{1}{2^{n^2+2n}}. \tag{5}$$

Combining (3), (4), and (5) gives

$$\begin{aligned} \gamma_n \frac{1}{2^n} = t_n &\leqslant \left(V_n \ \frac{b_n^{m_n}(1-b_n)^{n+2-m_n}}{m_n^{m_n}(n+2-m_n)^{n+2-m_n}}\right)^{1/(n+2)} \\ &\leqslant (\Delta_{n+2} \ \frac{b_n^{m_n}(1-b_n)^{n+2-m_n}}{m_n^{m_n}(n+2-m_n)^{n+2-m_n}}\Bigg)^{1/(n+2)} \\ &\leqslant \left(2 \text{ const } (n+2)^{n+2+1/4} \ \frac{1}{2^{n^2+2n}} \ \frac{b_n^{m_n}(1-b_n)^{n+2-m_n}}{m_n^{m_n}(n+2-m_n)^{n+2-m_n}}\right)^{1/(n+2)} \end{aligned}$$

for all sufficiently large $n \in \mathbb{N}$. By (2) it follows that

$$\begin{aligned} 0 < \frac{1-\eta}{1+\eta} &\leqslant \gamma = \liminf_{n \in L} \gamma_n \\ &\leqslant \liminf_{n \in L} \left((n+2)^{n+2} \frac{b_n^{m_n}(1-b_n)^{n+2-m_n}}{m_n^{m_n}(n+2-m_n)^{n+2-m_n}}\right)^{1/(n+2)} \\ &= \liminf_{n \in L} \frac{b_n^{m_n/(n+2)}(1-b_n)^{1-m_n/(n+2)}}{(m_n/(n+2))^{m_n/(n+2)}\,(1-m_n/(n+2))^{1-m_n/(n+2)}} \\ &\leqslant \liminf_{n \in L} \frac{b_n^b(1-b_n)^{1-b}}{b^b(1-b)^{1-b}}. \end{aligned}$$

We consider the function $\phi(x) := x^b(1-x)^{1-b}$, $x \in [0, 1]$, which is strictly increasing in $[0, b]$ and strictly decreasing in $[b, 1]$ with

$\phi(0)=\phi(1)=0$. There exist at most two solutions $0<m(b,\gamma)\leqslant b\leqslant M(b,\gamma)<1$ of $\phi(x)=\gamma b^b(1-b)^{1-b}>0$. In case $\gamma=1$, it follows that $m(b,\gamma)=b=M(b,\gamma)$.

From the inequality stated above, we obtain

$$0<m(b,\gamma)\leqslant \liminf_{n\in L} b_n\leqslant \limsup_{n\in L} b_n\leqslant M(b,\gamma)<1.$$

2. Let $\theta<\lambda$ be given. Then we may choose some $\alpha\in(0,1)$ such that $\limsup_{n\in L}(E_n/E_{[\alpha n]})\geqslant\theta$. Further, since $\lim_{n\in L}E_n^{1/n}=1$, we may choose a sequence of positive numbers $(\delta_n)_{n\in\mathbb{N}}$ such that $\lim_{n\in\mathbb{N}}\delta_n^{1/n}=1$ and $\lim_{n\in L}(\delta_n/E_n)=0$.

By ([7, Theorem 1]), there exists a sequence of polynomials $p_n\in P_n$, $n\in\mathbb{N}$, such that

$$\limsup_{n\in\mathbb{N}}\|f-p_n\|_K^{1/n}<1 \qquad \text{for each compact set} \quad K\subset A,$$

and

$$\|f-p_n\|_{[-1,1]}\leqslant E_{[\alpha n]}+\delta_n \qquad \text{for all} \quad n\in\mathbb{N}_0.$$

In view of (1), we obtain that for every compact set $K\subset A$,

$$\begin{aligned} E_n^2 &= \sum_{j=1}^{n+2}\mu_n(x_{n,j})\,(f-p_n^*)(x_{n,j})\,(f-p_n^*)(x_{n,j})\\ &= \sum_{j=1}^{n+2}\mu_n(x_{n,j})\,(f-p_n^*)(x_{n,j})(f-p_n)(x_{n,j})\\ &\leqslant E_n\{\mu_n(K)\,\|f-p_n\|_K+(1-\mu_n(K))\,\|f-p_n\|_{[-1,1]}\}\\ &\leqslant E_n\{\mu_n(K)\,\|f-p_n\|_K+(1-\mu_n(K))\,(E_{[\alpha n]}+\delta_n)\}. \end{aligned}$$

This yields

$$\mu_n(K)\leqslant 1-\frac{E_n-\|f-p_n\|_K\,\mu_n(K)}{E_{[\alpha n]}+\delta_n},$$

and, by the properties of $(p_n)_n$ and our choice of $(\delta_n)_{n\in\mathbb{N}}$, we obtain $\liminf_{n\in L}\mu_n(K)\leqslant 1-\theta$. Since $\theta<\lambda$ was arbitrary, it follows that

$$\liminf_{n\in L}\mu_n(K)\leqslant 1-\lambda$$

holds for every compact set $K\subset A$.

3. Applying part 1 of the proof to $B=K$, we get

$$m(\mu_{[-1,1]}(K),\gamma)\leqslant\liminf_{n\in L}\mu_n(K)\leqslant 1-\lambda$$

for every compact set $K\subset A$ with $\mu_{[-1,1]}(K)\in(0,1)$.

(a) Let $\eta = 0$. Then $\gamma = 1$, which yields

$$\mu_{[-1,1]}(K) = m(\mu_{[-1,1]}(K), 1) \leqslant 1 - \lambda$$

for every compact set $K \subset A$ with $\mu_{[-1,1]}(K) \in (0,1)$. It follows that $\mu_{[-1,1]}(A) \leqslant 1 - \lambda$.

(b) Let $\lambda = 1$ and assume that $A \neq \emptyset$. Then there exists a compact subset $K \subset A$ with $\mu_{[-1,1]}(K) \in (0,1)$, which implies a contradiction:

$$0 < m(\mu_{[-1,1]}(K), \gamma) \leqslant 1 - \lambda = 0.$$

Proof of Theorem 2. 1. Let $\liminf_{k \in \mathbb{N}} E_{n_k}^{1/n_k} < 1$.

We show that there exists a subsequence L of $(n_k)_k$ such that

$$\lim_{n \in L} E_n^{1/n} = 1 \qquad \text{and} \qquad \lim_{n \in L} \frac{E_{n+1}}{E_n} = 0.$$

Since $\limsup_{n \in \mathbb{N}} E_n^{1/n} = 1$, it follows that $\limsup_{k \in \mathbb{N}} E_{n_k}^{1/n_k} = 1$. Thus, we may choose a subsequence $(n_{k_j})_{j \in \mathbb{N}}$ of $(n_k)_k$ such that for all $j \in \mathbb{N}$,

$$E_{n_{k_j}}^{1/n_{k_j}} \geqslant 1 - \frac{1}{j} \qquad \text{and} \qquad \left(\frac{1}{j}\right)^{1/n_{k_j}} > \left(1 - \frac{1}{j}\right)^{1-1/\rho}.$$

By an inductive argument we will show that for every j one of the following two alternatives must hold:

— there exists some $l_j \in \{k_j, ..., k_{j+1}\}$ such that

$$\frac{E_{n_{l_j}+1}}{E_{n_{l_j}}} \leqslant \frac{1}{j} \qquad \text{and} \qquad E_{n_{l_j}}^{1/n_{l_j}} \geqslant 1 - \frac{1}{j}, \tag{6}$$

— for all $k \in \{k_j, ..., k_{j+1}\}$ we have

$$E_{n_k}^{1/n_k} \geqslant 1 - \frac{1}{j}. \tag{7}$$

To prove this, we let j be fixed and observe that (7) holds for $k = k_j$.

Suppose that (6) does not hold for $l_j = k_j$. Then we must have

$$\frac{E_{n_{k_j}+1}}{E_{n_{k_j}}} > \frac{1}{j},$$

and therefore

$$E_{n_{k_j+1}}^{1/(n_{k_j+1})} \geqslant E_{n_{k_j}+1}^{1/(n_{k_j}+1)} \geqslant \left(\frac{1}{j} E_{n_{k_j}}\right)^{1/(n_{k_j}+1)}$$

$$\geqslant \left(1 - \frac{1}{j}\right)^{1-1/\rho} \left(1 - \frac{1}{j}\right)^{n_{k_j}/(n_{k_j}+1)}$$

$$\geqslant \left(1 - \frac{1}{j}\right)^{1-1/\rho} \left(1 - \frac{1}{j}\right)^{1/\rho} = 1 - \frac{1}{j}.$$

Thus, (7) holds for $k = k_j + 1$. Now, if we suppose that (6) does not hold for $l_j = k_j + 1$, it follows in the same way that (7) holds for $k = k_j + 2$. Proceeding in this way, we obtain that (6) holds for some $l_j \in \{k_j, ..., k_{j+1}\}$ or (7) holds for all $k \in \{k_j, ..., k_{j+1}\}$.

If the first alternative holds only for finitely many j, then it follows from (7) that $\lim_{k \in \mathbb{N}} E_{n_k}^{1/n_k} = 1$, which contradicts our assumption. Therefore, it must hold for infinitely many j, and we can choose a subsequence $L = (n_{l_j})_j$ with the desired properties.

Choosing $\alpha = 1/\rho$, we see that

$$\lambda := \sup_{\alpha \in (0,1)} \limsup_{n \in L} \frac{E_n}{E_{[\alpha n]}} = 1$$

and our statement follows from part (b) of Theorem 1.

2. Let $\lim_{k \in \mathbb{N}} E_{n_k}^{1/n_k} = 1$.

Then we may apply part (b) of Theorem 1 to a subsequence L of $(n_k)_k$ with

$$\lim_{n \in L} \frac{E_{n+1}}{E_n} = \liminf_{k \to \infty} \frac{E_{n_k+1}}{E_{n_k}}.$$

If we choose $\alpha \in (1/\rho, 1)$, it follows immediately that

$$\lambda := \sup_{\alpha \in (0,1)} \limsup_{n \in L} \frac{E_n}{E_{[\alpha n]}} = 1,$$

and it remains to show that

$$\liminf_{k \to \infty} \frac{E_{n_k+1}}{E_{n_k}} < 1.$$

Suppose that $\liminf_{k\to\infty}(E_{n_k+1}/E_{n_k})=1$. Then, for some arbitrary $\gamma\in(0,\beta)$, there exists some $k_1\in\mathbb{N}$ such that for all $k\geqslant k_1$,

$$\frac{E_{n_k+1}}{E_{n_k}}\geqslant\left(\frac{1}{\rho}\right)^{\gamma}.$$

Since $n_{k+1}/n_k\geqslant\rho>1$, we have $n_k\geqslant\rho^k n_0\geqslant\rho^k$, and thus $k\leqslant\log(n_k)/\log(\rho)$ for all $k\in\mathbb{N}$. It follows that there exist positive constants M_1, M_2, such that for all $k\geqslant k_1$ we have

$$\begin{aligned} M_1\left(\frac{1}{n_k}\right)^{\beta} &\geqslant E_{n_k}=E_{n_0}\prod_{j=0}^{k-1}\frac{E_{n_{j+1}}}{E_{n_j}}\\ &\geqslant M_2\left(\frac{1}{\rho}\right)^{\gamma k}\geqslant M_2\left(\frac{1}{\rho}\right)^{\gamma(\log(n_k)/\log(\rho))}=M_2\left(\frac{1}{n_k}\right)^{\gamma}, \end{aligned}$$

which implies a contradiction.

Proof of Theorem 3. For every $n\in\mathbb{N}$ we have

$$\begin{aligned} E_n=E_n(f)&=\|\operatorname{Re} f-\operatorname{Re} p_n^*(f)+i(\operatorname{Im} f-\operatorname{Im} p_n^*(f))\|_{[-1,1]}\\ &\geqslant\max\{E_n(\operatorname{Re} f), E_n(\operatorname{Im} f)\} \end{aligned}$$

and

$$\begin{aligned} E_n=E_n(f)&\leqslant\|\operatorname{Re} f-p_n^*(\operatorname{Re} f)+i(\operatorname{Im} f-p_n^*(\operatorname{Im} f))\|_{[-1,1]}\\ &\leqslant 2^{1/2}\max\{E_n(\operatorname{Re} f), E_n(\operatorname{Im} f)\}. \end{aligned}$$

The function f is regular at some point $x_0\in[-1,1]$ if and only if its real part $\operatorname{Re} f$ and its imaginary part $\operatorname{Im} f$ are both regular at x_0. Without loss of generality, let L' be a subsequence of L such that $\max\{E_n(\operatorname{Re} f), E_n(\operatorname{Im} f)\}=E_n(\operatorname{Re} f)$ for all $n\in L'$. It follows that

$$\frac{E_n}{2^{1/2}}\leqslant E_n(\operatorname{Re} f)\leqslant E_n\qquad\text{for all}\quad n\in L',$$

and therefore we have

$$\lim_{n\in L'} E_n(\operatorname{Re} f)^{1/n}=1,$$

$$\lim_{n\in L'}\frac{E_{n+1}(\operatorname{Re} f)}{E_n(\operatorname{Re} f)}\leqslant 2^{1/2}\lim_{n\in L}\frac{E_{n+1}}{E_n}=0,$$

and

$$\sup_{\alpha \in (0,1)} \limsup_{n \in L'} \frac{E_n(\operatorname{Re} f)}{E_{[\alpha n]}(\operatorname{Re} f)} \geqslant \frac{1}{2^{1/2}} \sup_{\alpha \in (0,1)} \limsup_{n \in L} \frac{E_n}{E_{[\alpha n]}} = \frac{1}{2^{1/2}} \lambda.$$

Our statement follows if we apply part (a) of Theorem 1 to the function $\operatorname{Re} f$ and the subsequence L'.

Proof of Theorem 4. 1. We first consider the case $r=1$, i.e., $C_{1/r} = [-1, 1]$.

We assume that f is regular at some point $x_0 \in [-1, 1]$, which implies that f is regular in a closed neighbourhood $U_t(x_0) := \{z \in \mathbb{C} : |z - x_0| \leqslant t\}$, $t > 0$, of x_0.

It follows from our assumptions that there exists some $q < 1$ such that

$$(\|f - p^*_{n+1}\|_{[-1,1]})^{1/(n+1)} \leqslant q \tag{8}$$

holds for all sufficiently large $n \geqslant n_1$, $n \in L$.

By the Bernstein–Walsh inequality ([9, p. 70]), we have

$$|p^*_{n+1}(z)| \leqslant \|p^*_{n+1}\|_{[-1,1]} \exp((n+1)\, g(z))$$

for all $z \in \mathbb{C}$. Since f is bounded in $U_t(x_0)$, one can see that for all sufficiently large $n \geqslant n_2$, $n \in L$,

$$|f(z) - p^*_{n+1}(z)|^{1/(n+1)} \leqslant (|f(z)| + |p^*_{n+1}(z)|)^{1/(n+1)} \leqslant 2 \exp(g(z)) \tag{9}$$

holds for all $z \in U_t(x_0)$.

We put $I := [-1, 1] \cap [x_0 - t/2, x_0 + t/2]$ and denote by u the solution of the Dirichlet problem in $U_t(x_0) \backslash I$ with boundary values

$$u(z) = \begin{cases} \log(2 \exp(g(z))), & \text{for all } \; z \in \{z : |z - x_0| = t\} \\ \log(q) < 0, & \text{for all } \; z \in I \end{cases}.$$

Since u is continuous, there exists some $m < 0$ and some closed neighbourhood $U_s(x_0)$, $0 < s < t$, such that $u(z) \leqslant m < 0$ for all $z \in U_s(x_0)$.

The functions $(1/(n+1)) \log |f(z) - p^*_{n+1}(z)|$ are subharmonic in $U_t(x_0)$, and by (8) and (9) we obtain

$$\frac{1}{n+1} \log |f(z) - p^*_{n+1}(z)| \leqslant u(z)$$

for all $z \in \{z : |z - x_0| = r\} \cup I$ and all $n \geqslant n_0 := \max\{n_1, n_2\}$, $n \in L$.

It follows from majorization principles for subharmonic functions that

$$|f(z)-p_{n+1}^*(z)|^{1/(n+1)} \leqslant \exp(u(z))$$

holds for all $z \in U_t(x_0)$ and all $n \geqslant n_0$, $n \in L$. Thus, we have

$$\|f-p_{n+1}^*\|_{U_s(x_0)}^{1/(n+1)} \leqslant \exp(m) < 1,$$

such that $(p_{n+1}^*)_{n \in L}$ converges to f uniformly on $K := [-1, 1] \cup U_s(x_0)$.

In particular, the sequence

$$p_{n+1}^*(z) = a_{n+1} z^{n+1} + \cdots, \qquad n \in L,$$

is uniformly bounded on K. Note that for sufficiently large $n \in L$ we have $E_n > E_{n+1}$, which implies that $a_{n+1} \neq 0$. If $\operatorname{cap}(K)$ denotes the logarithmic capacity or Chebychev constant of K, then $\operatorname{cap}(K) > \operatorname{cap}([-1, 1]) = 1/2$. Since

$$\operatorname{cap}(K) \leqslant \liminf_{n \in L} \left(\frac{\|p_{n+1}^*\|_K}{|a_{n+1}|} \right)^{1/(n+1)} = \liminf_{n \in L} \frac{1}{|a_{n+1}|^{1/(n+1)}},$$

we get

$$\limsup_{k \to \infty} |a_{n+1}|^{1/(n+1)} \leqslant \frac{1}{\operatorname{cap}(K)} < \frac{1}{\operatorname{cap}([-1, 1])} = 2.$$

Let $T_n(x) := x^n + \ldots$, $n \in \mathbb{N}$, denote the nth Chebychev-polynomial of the set $[-1, 1]$. Then $\|T_n\|_{[-1, 1]} = 1/2^{n-1}$, and we obtain a contradiction:

$$\begin{aligned} 1 = \limsup_{n \in L} E_n^{1/n} &\leqslant \limsup_{n \in L} \|f - p_{n+1}^* + a_{n+1} T_{n+1}\|_{[-1, 1]}^{1/n} \\ &\leqslant \limsup_{n \in L} (\|f-p_{n+1}^*\|_{[-1, 1]} + \|a_{n+1} T_{n+1}\|_{[-1, 1]})^{1/n} < 1. \end{aligned}$$

2. The idea of the proof for $r \in (0, 1)$ is essentially the same as for $r = 1$ such that we give only the most important steps of it.

We assume that f is regular at some point $z_0 \in C_{1/r}$.

From results on maximal convergence ([9, p. 90]) it follows that

$$\limsup_{n \in \mathbb{N}} \|f - p_n^*\|_Q^{1/n} \leqslant \|r \exp(g)\|_Q$$

for every compact set $Q \subset \{z : g(z) < -\log(r)\}$. Since we have

$$\limsup_{n \in L} \|f - p_{n+1}^*\|_{[-1, 1]}^{1/(n+1)} < r,$$

one can show by principles of harmonic majorization that

$$\limsup_{n \in L} \| f - p^*_{n+1} \|_Q^{1/(n+1)} < \| r \exp(g) \|_Q$$

holds for every compact set $Q \subset \{z: g(z) < -\log(r)\}$.

By ([8, Theorem 5]), there exists a neighbourhood $U(z_0)$ of z_0 such that $(p^*_{n+1})_{n \in L}$ converges to f locally uniformly in $\{z: g(z) < -\log(r)\} \cup U(z_0)$. If we put $K := \overline{\{z: g(z) < -\log(r)\} \cup U(z_0)}$, then, by the Bernstein–Walsh Lemma,

$$\limsup_{n \in L} \| p^*_{n+1} \|_K^{1/(n+1)} \leqslant 1.$$

Since $\operatorname{cap}(K) > \operatorname{cap}(C_{1/r}) = 1/2r$, a contradiction is obtained in the same way as in part 1 of the proof.

Proof of Theorem 5. We apply Theorem 4 to a suitable subsequence L of $(n_k)_k$. It is easy to see that

$$\limsup_{k \in \mathbb{N}} E_{n_k}^{1/n_k} = \limsup_{n \in \mathbb{N}} E_n^{1/n} = r.$$

Hence, we may choose a subsequence L of $(n_k)_k$ such that $\lim_{n \in L} E_n^{1/n} = r$. By the properties of $(n_k)_k$, and since $r \in (0, 1)$, we obtain

$$\limsup_{n \in L} E_{n+1}^{1/(n+1)} \leqslant \limsup_{k \to \infty} E_{n_k+1}^{1/(n_k+1)} = \limsup_{k \to \infty} E_{n_{k+1}}^{1/(n_k+1)}$$

$$= \limsup_{k \to \infty} (E_{n_{k+1}}^{1/n_{k+1}})^{n_{k+1}/(n_k+1)} \leqslant r^\rho < r,$$

which proves our statement.

ACKNOWLEDGMENT

I am thankful to Professor Gaier for his helpful comments.

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